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Mean Field Analysis of Hamiltonian SU(2) Lattice Gauge Theory*

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ABSTRACT

We study two types of vacuum wave-functions for the SU(2) lattice gauge theory in 3+1 dimensions. One trial function is based on a mean-plaquette ansatz which is analysed by employing a 3-dimensional Monte Carlo program for an Euclidean SU(2) gauge theory. The second is based on a mean-link ansatz which is projected onto its gauge-invariant component. Its numerical analysis is more complicated and involves a chiral SU(2) problem with an heat-kernel action in 3-dimensions. The results of both trial functions lead to similar ground-state energies.

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1. INTRODUCTION

We study here two types of vacuum wave-functions for the SU(2) gauge theory in 3+1 dimensions. Both types of wave functions rely on a mean-field ansatz, however one uses a mean-plaquette description whereas the other employs a mean-link method.

The SU(2) problem is defined by the Hamiltonian of Kogut and Susskind[1] on a 3-dimensional lattice in space. By using a specific ansatz for the ground-state wave-function one reduces the problem to 3-dimensional Euclidean statistical-mechanics. The norm of the wave-function becomes a partition-function and the vacuum energy turns into a certain thermodynamic quantity which may be evaluated by numerical methods.

Such an approach has been applied to the SU(2) problem in 2+1 dimensions[2] where the mean-plaquette ansatz allows for an analytic solution. This is based on the fact that on a 2-dimensional spacial lattice one may transform the original link variables into independent plaquette variables. This is no longer true in 3-dimensions. Therefore the mean-plaquette ansatz which we discuss in Section 2 necessitates a Monte-Carlo calculation[3,4] of the 3-dimensional SU(2) gauge problem.

The second ansatz which we use is the gauge-invariant mean-link method of Ref. 5. Projecting a wave-function of the link variables onto its gauge-invariant sector leads to statistical mechanics of the gauge-variables defined on the lattice vertices. The original type of ansatz for the link wave-function fixes the kernel of the partition function. In our case it is the heat-kernel action for chiral SU(2) in 3-dimensions, which we discuss in Appendix A. Its physical consequences for our problem as well as the comparison between this ansatz and the previous one are discussed in Section 3. The group-theory that we need for our calculation is developed in Appendix B.

The results are summarized in Section 4 where we discuss also possible modifications which may improve the structure of the wave function in the weak-coupling regime by introducing long range correlations into it.

2. MEAN-PLAQUETTE METHOD

The SU(2) Hamiltonian is chosen to be[1]

$$H = \frac{g^2}{2} \sum_{\ell} \vec{E}_{\ell}^{\,2} + \frac{2}{g^2} \sum_{p} \left(2 - \mathrm{tr}\, U_p\right) \qquad (1)$$

where the basic variable is the SU(2) group-element (2×2 matrix) associated with each link, U_ℓ, and U_p is its path-ordered product along a plaquette: $U_p=U_1U_2U_3^{-1}U_4^{-1}$. $\vec{E}_\ell$ is the color-electric field which is a vector in color-space and a component (in the ℓ-direction) of a vector in real space. It obeys the relation

$$[E_\ell^a, U_{\ell'}] = \frac{\sigma^a}{2} U_\ell \, \delta_{\ell\ell'} \tag{2}$$

The simplest gauge-invariant ansatz for the vacuum, is given by

$$\psi = Z^{-1/2} \exp\left(\frac{\beta}{2} \sum_p \mathrm{tr}\, U_p\right) \tag{3}$$

The normalization factor Z is the partition function of a 3-dimensional SU(2) problem based on the Wilson-action:

$$Z = \int \mathcal{D}U_\ell \; e^{\beta \sum_p \mathrm{tr}\, U_p} \tag{4}$$

Following Arisue et al.[2] we note that the energy density can now be written as

$$\mathcal{E}(g^2,\beta) = \frac{\langle H\rangle_\psi}{N_p} = \left(\frac{3g^2}{8}\beta - \frac{2}{g^2}\right)\langle \mathrm{tr}\, U_p\rangle_Z + \frac{4}{g^2} \tag{5}$$

where N_p is the number of plaquettes of the three dimensional lattice. $\langle H\rangle_\psi$ in Eq. (5) denotes the quantum-mechanical average of H in the state ψ defined by Eq. (3), while $\langle \mathrm{tr}\, U_p\rangle_Z$ stands for the statistical average of $\mathrm{tr}\, U_p$ using Z of Eq. (4). The latter can be calculated numerically by Monte-Carlo techniques and thus one can evaluate $\mathcal{E}(g^2,\beta)$. Requiring

$$\frac{\partial \mathcal{E}(g^2,\beta)}{\partial \beta} = 0 \qquad\qquad \frac{\partial^2 \mathcal{E}}{\partial \beta^2} > 0 \tag{6}$$

we find the parameter $\beta(g^2)$ which minimizes the vacuum energy.

The SU(2) problem in 3-dimensions, as defined by Eq. (4), was studied by d'Hoker[3] who found the high-β behaviour

$$\langle \mathrm{tr}\, U_p \rangle \approx 2 - \frac{1}{\beta} - \frac{0.35 \pm 0.2}{2\beta^2} \qquad \text{for } \beta \to \infty \tag{7}$$

Inserting it into Eq. (5) we obtain the weak-coupling limit of our ansatz

$$\beta^2 \approx \frac{8}{3g^4} \qquad \mathcal{E} \to \sqrt{6} \qquad \text{as } g \to 0 \tag{8}$$

In the other extreme limit, the strong coupling where $g \to \infty$ and $\beta \to 0$, one may use for comparison the single-plaquette problem which yields

$$\langle \mathrm{tr}\, U_p \rangle = \frac{2\, I_2(2\beta)}{I_1(2\beta)} \to \beta \qquad \text{as } \beta \to 0 \tag{9}$$

and therefore

$$\beta \approx \frac{8}{3g^4} \qquad \mathcal{E} \to \frac{4}{g^2} - \frac{8}{3g^6} \qquad \text{as } g \to \infty \tag{10}$$

We have evaluated $\mathcal{E}(g^2,\beta)$ by using the Monte-Carlo program developed by M. Creutz[4] for the SU(2) lattice gauge theory. We have used an 8^3 lattice and changed β in steps of 0.1. In order to obtain stable results it was sufficient to use 40 iterations for each value of β; results remained the same (within 0.5%) after 300 iterations. In Fig. 1 we display the final results for $\mathcal{E}$ after the minimization procedure was carried out. For comparison we show the results for a single plaquette problem [i.e. using Eq. (9) for $\langle \mathrm{tr}\, U_p \rangle$] and we see that the departure between the two becomes appreciable below $g^2 \leq 2.4$.

The ansatz that we have used here for ψ, Eq. (3), is of course best suited for the strong coupling region. The analysis of the SU(2) model in 2+1 dimensions[2] has shown that this ansatz is very stable: adding to the exponent terms which involve two neighboring plaquettes one finds that they acquire minute coefficients. It is however clear that the ansatz does not display the correct physics in the weak coupling region. It is in fact too strongly confining. Calculating a spacial Wilson-loop one would be led to the same result as that of a Wilson-loop in the 3-dimensional theory of Eq. (4) with the relation (8) between β and g^2 near $g \to 0$. As a result it will vanish like a power as $g \to 0$ whereas one would expect it to vanish like an essential singularity at this point in 3+1 dimensions. The true vacuum will therefore require a more complicated expression in the weak coupling region. We will discuss possible modifications in Section 5.

3. MEAN LINK METHOD

Our second mean-field method[5] is based on the use of a link wave function $\phi(U_\ell)$. Taking the product over all links

$$\psi_{ML} = \prod_\ell \phi(U_\ell) \tag{11}$$

we obtain a state which is not gauge invariant. Projecting this state onto its gauge invariant part, $P\psi$, we obtain a trial wave function for the vacuum. Since $P^2=P$ its norm is given by

$$Z = \int \mathcal{D}U_\ell \, \psi^*_{ML} P \psi_{ML} = \int \mathcal{D}G_i \prod_\ell \int dU_\ell \, \phi^*(U_\ell)\phi(G_{\ell-}^{-1} U_\ell G_{\ell+}) \tag{12}$$

G_i are the group elements which are introduced at every vertex of the lattice. The projection operation involves the rotation of every link element U_ℓ by the two G_i which are located at its two end points $i=\ell_\pm$.

The wave function we use[5] is

$$\phi = \sum_j e^{-\frac{j(j+1)}{\beta}} (2j+1)\, \chi_j(U_\ell) \tag{13}$$

which is the analog of a discrete-Gaussian, or the Villain approximation, in the U(1) theory. χ_j is the character of the j-th representation of SU(2). Inserting it into Eq. (12) we find

$$Z = \int \mathcal{D}G_i \prod_\ell K(G_{\ell-}^{-1} G_{\ell+})$$
$$K = \sum_j e^{-\frac{2j(j+1)}{\beta}} (2j+1)\, \chi_j \tag{14}$$

Our choice of ϕ led to K which is of the same nature. One important property of this K is that it is a positive definite function. Therefore Z defines a partition function also known as the heat-kernel problem.[6] This name stems from the fact that K obeys the diffusion equation

$$\frac{\partial K}{\partial \beta} = \frac{2}{\beta^2} \vec{E}^2 K \tag{15}$$

where $\vec{E}^2$ is the Laplace-Beltrami operator on the group.

Z of Eq. (14) is a global-symmetry problem and therefore has a completely different behaviour from the local-symmetry problem of Eq. (4). The partition function of the previous section defined a problem with no phase transition, whereas the one used here has a continuous phase transition. Although this transition is very smooth--as exhibited in Appendix A--it causes a problem because it introduces a spurious phase transition into our analysis.

In order to obtain $\beta(g)$ we have to calculate the expectation value of H. This is done by representing every term which appears in H as an operator in the statistical mechanics of Z. It is straightforward to obtain

$$\frac{\langle\psi|\vec{E}_\ell^{\,2} P|\psi\rangle}{\langle\psi|P|\psi\rangle} = \left\langle \frac{\beta^2}{2}\frac{\partial}{\partial\beta} \ln K(G_{\ell-}^{-1} G_{\ell+}) \right\rangle_Z \tag{16}$$

which is a consequence of Eq. (15). The expectation value of $\mathrm{tr}\, U_p = \chi_{1/2}(U_p)$ is more complicated. It involves integrals of the form (see Fig. 2 for notation).

$$\begin{aligned}\int dU_1 \ldots dU_4\ & \chi_{j_1'}(U_1)\,\chi_{j_2'}(U_2)\,\chi_{j_3'}(U_3)\,\chi_{j_4'}(U_4)\,\chi_{1/2}(U_1U_2U_3^{-1}U_4^{-1})\cdot \\ & \cdot\chi_{j_1}(G_1^{-1}U_1G_2)\,\chi_{j_2}(G_2^{-1}U_2G_3)\,\chi_{j_3}(G_4^{-1}U_3G_3)\,\chi_{j_4}(G_1^{-1}U_4G_4)\end{aligned} \tag{17}$$

Straightforward calculation shows that this integral can be expressed in the form

$$\mathrm{tr}\ V_{j_1j_1'}(G_2G_1^{-1})\,V_{j_2j_2'}(G_3G_2^{-1})\,V_{j_3j_3'}(G_4G_3^{-1})\,V_{j_4j_4'}(G_1G_4^{-1}) \tag{18}$$

where V is a 2×2 matrix defined by

$$\begin{aligned}(2j+1)(2j'+1)\,V_{jj'}(G) = {} & \delta(j'-j-\tfrac{1}{2})\left[(j+1)\chi_j(G)\mathbb{1} + i\rho_j(G)\,\vec{g}\cdot\vec{\sigma}\right] \\ & + \delta(j'-j+\tfrac{1}{2})\left[j\chi_j(G)\mathbb{1} - i\rho_j(G)\,\vec{g}\cdot\vec{\sigma}\right]\end{aligned} \tag{19}$$

In the last expression we use the notation explained in Appendix B where we develop the group-theoretical basis needed for this calculation. Incorporating the weights $e^{-j(j+1)/\beta}$ associated with $\chi_j(U_\ell)$ in our wave function (13) we are led to define the 2×2 matrix

$$A(G) = \sum_{jj'} e^{-\frac{j'(j'+1)}{\beta}} e^{-\frac{j(j+1)}{\beta}} (2j+1)(2j'+1) V_{jj'}(G) / K(G) \quad (20)$$

This allows us to express the quantum-mechanical matrix element of $\chi_{1/2}(U_1U_2U_3^{-1}U_4^{-1})$ as the statistical average

$$\frac{\langle\psi|\chi_{1/2}(U_1U_2U_3^{-1}U_4^{-1})P|\psi\rangle}{\langle\psi|P|\psi\rangle} = \langle \mathrm{tr}\, A(G_2G_1^{-1})A(G_3G_2^{-1})A(G_4G_3^{-1})A(G_1G_4^{-1})\rangle_Z \quad (21)$$

using the partition function Z of Eq. (12).

We are now at the stage in which we can carry out the calculation. Using the Monte Carlo procedure for the heat-kernel problem, Eq. (12), we obtain thermalized configurations of the vertex group elements $\{G_i\}$ which can be used to calculate the statistical averages of Eq. (16) and Eq. (21). We have carried out the Monte Carlo simulation (explained in Appendix A) on a 5^3 lattice using Y, the icosahedral subgroup of SU(2).[7] The use of this 120-element subgroup is essential when one encounters such complicated expressions as Eq. (20). We found that the plaquette-term of Eq. (21) converges quickly to its equilibrium limit but the electric field term, Eq. (16), exhibits large fluctuations. This made it necessary to perform 4000 iterations per value of β. After spanning β by steps of 0.1 we performed a minimization calculation for the energy thus leading to $\beta(g)$ which is displayed in Fig. 3 and $\mathcal{E}(g)$ shown in Fig. 4.

Figure 3 shows the correspondence between β and $2/g^2$ in two phases of the icosahedral heat-kernel problem. Below $\beta_c \lesssim 0.95$ we find the strong-coupling phase. The kink at β_c is a clear indication of the phase transition of the underlying chiral model. In the range $0.95<\beta<2.9$ we cover $0.8 \lesssim 2/g^2 \lesssim 1.5$ thus leading us into the weak-coupling domain. $\beta_F \simeq 2.9$ is the point where the icosahedral subgroup freezes out and stops having the same physical features as the continuous SU(2) theory (see Appendix A). Hence this calculation is not applicable at $2/g^2>1.5$.

In Fig. 4 we display the resulting energy alongside with the results of the mean-plaquette calculation of the previous section. We see that in the strong-coupling region both calculations lead to the same results. Turning into the weak-coupling region the mean-link method lags behind but at $2/g^2 \simeq 1.4$ it starts taking over. Since the calculation is no longer applicable above $2/g^2>1.5$ we estimated its behaviour by extrapolating fits to our numerical calculations of Eq. (16) and (21) from $2<\beta<2.8$. Our conclusion is that the mean-link result follows the mean-plaquette one for a wide range of $2/g^2$.

4. SUMMARY AND OUTLOOK

The comparison between the mean-plaquette ansatz and the mean-link method carried out in this paper came out in favor of the mean-plaquette one. Not only is it much easier to calculate with, but its energy turns out to be lower than the mean-link one for most of the range over which we can compare them. Moreover it does not suffer from a phase transition in β therefore we are assured of its confining nature for all g^2.

It is nonetheless interesting to look at the mean-link ansatz because the latter can be readily generalized to include quark fields as dynamical variables in the Hamiltonian. Figure 4 shows that the underlying chiral model crosses through a phase transition at the point $g^2 \simeq 2.5$. This value corresponds to the location of the cross-over region according to the large order perturbation calculation of Kogut and Shigemitsu.[8] Hence it seems that the occurrence of the phase transition at this particular point is not fortuitous. Nonetheless it is of course an artifact of the mean-link method which should not survive in an improved analysis.

The way the mean-link method may be improved was proposed in Ref. 5. Operating on ψ_{ML} of Eq. (11) with

$$\psi_{ML} \rightarrow e^{-\sum_{\ell\ell'} \vec{J}(\ell)\,\Delta(\ell,\ell')\,\vec{J}(\ell')}\;\psi_{ML} \qquad (22)$$

one arrives at a wave-function which includes correlations between different links. Projecting out the gauge-invariant component one obtains a much more complicated chiral model. This was shown[5] to lead to the correct qualitative behaviour for the U(1) theory in 2+1 dimensions where $\Delta(\ell\ell')$ became a propagator with a mass which vanishes in weak coupling with an essential singularity. To repeat the same analytic calculation for the SU(2) problem seems to be impossible. Therefore one will have to work with a series of $\Delta(\ell\ell')$ which lead to gradually more complicated calculations as larger separations between ℓ and ℓ' are allowed. Including such larger separations one may hope to push the phase transition point further into the weak-coupling regime so that the whole transition region fits into one phase of the underlying chiral model.[5]

The mean-plaquette ansatz has the confining nature of the 2+1 dimensional theory and has therefore to be modified considerably to show the right behaviour in 3+1 dimensions. Following the lesson of the 2+1 dimensional analysis[2] one may conclude that adding nearest neighbor plaquettes will slightly improve the behaviour. We would like to point out that one can be more ambitious and study a wave function which includes long range correlations in its exponent. Our candidate wave function would be

$$\psi = Z^{-1/2} \exp\left\{ \frac{\beta}{2} \sum_{p} \mathrm{tr}\, U_p + \frac{1}{2} \sum_{pq} \mathrm{tr}\, U_p \, \Delta(p,q) \, \mathrm{tr}\, U_q \right\} \qquad (23)$$

The point to be stressed is that this ansatz is amenable to numerical calculations following the line of reasoning of Section 2. The reason is that the bilinear term in the exponent can be rewritten in terms of an auxiliary plaquette field ϕ_p:

$$\int \mathcal{D}\varphi_p \exp\left\{-\frac{1}{2}\sum_{pq} \varphi_p \Delta^{-1}(p,q)\varphi_q + \sum_p \mathrm{tr}\, U_p \varphi_p\right\} \sim \sim \exp\left\{\frac{1}{2}\sum_{pq} \mathrm{tr}\, U_p \Delta(p,q)\, \mathrm{tr}\, U_q\right\} \tag{24}$$

Hence, with the aid of this auxiliary field, ψ can be written in a form which involves only a single tr U_p for every plaquette. Equation (15) may then be generalized into a form which involves averages of terms like tr $U_p \exp(\sum_p \phi_p \text{ tr } U_p)$ in a distribution of ϕ goverened by the 3-dimensional propagator $\Delta^{-1}(p,q)$. ϕ_p may be regarded as a scalar field which is equivalent to a compound singlet structure of the original gauge fields. If the ansatz (24) describes correct features of the vacuum then the g^2 dependence of the mass in the propagator should reflect the β-function behaviour of the continuum SU(2) theory.

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APPENDIX A: CHIRAL SU(2) PROBLEM IN 3 DIMENSIONS[9]

We study a lattice model which possesses a global SU(2)×SU(2) symmetry. With each site i of a cubic lattice we associate an element of the SU(2) group G(i). The simplest action which has the correct symmetry character is given by the nearest-neighbor interaction

$$A_1 = \frac{\beta}{2} \sum_{i,\mu} \operatorname{tr}\left(G^{-1}(i)\, G(i+\hat{e}_\mu)\right) \tag{A1}$$

where i refers to the lattice site and $\hat{e}_\mu$ are the three independent unit vectors. Viewing G as a 2×2 matrix-element we realize that A_1 is defined in terms of the character of the fundamental representation. The kernel of the corresponding partition function

$$Z_1 = \int \mathcal{D}G_i \; e^{A_1} \tag{A2}$$

may be expanded in terms of all characters of the SU(2) group

$$e^{\frac{1}{2}\beta \chi_{1/2}(G)} = \sum_{j=0}^{\infty} \frac{2(2j+1)}{\beta}\, I_{2j+1}(\beta)\, \chi_j(G) \tag{A3}$$

where I_{2j+1} is the modified Bessel function of order 2j+1.

We wish to demonstrate that the heat-kernel action that we need for the analysis in Section 3 is a Villain approximation to A_1. For that purpose we note that the large β behaviour of the Bessel functions

$$I_{2j+1}(\beta) \underset{\beta\to\infty}{\longrightarrow} \frac{e^{\beta - \frac{1}{2\beta}}}{\sqrt{2\pi\beta}}\, e^{-\frac{2j(j+1)}{\beta}} \tag{A4}$$

leads to

$$e^{\frac{1}{2}\beta\chi_{1/2}(G)} \underset{\beta\to\infty}{\longrightarrow} \frac{2\, e^{\beta-\frac{1}{2\beta}}}{\beta\sqrt{2\pi\beta}} \sum_{j=0}^{\infty} e^{-\frac{2j(j+1)}{\beta}} (2j+1)\chi_j(G) \tag{A5}$$

Dropping the overall β dependent factor we are led to the heat-kernel action

$$e^{A_2} = \prod_{i,\mu} K\left(G^{-1}(i)\, G(i+\hat{e}_\mu)\right) \tag{A6}$$

where

$$K(G) = \sum_j e^{-\frac{2j(j+1)}{\beta}} (2j+1)\,\chi_j(G) \tag{A7}$$

The action A_1 can be easily studied by means of a Monte-Carlo simulation. The situation is different for the heat kernel action. First, the evaluation of the sum over all representations of SU(2) is of course impossible in practice. One must introduce a cutoff on the allowed values of the "angular momentum" j. For any given value of β we can terminate the j series if $e^{-2[j(j+1)]/\beta} \ll 1$. After the cutoff is introduced we will have to calculate rather complicated expressions for each link. Without further approximations we would need extremely large amounts of computer time to carry out the computation. Fortunately enough it is well known from Euclidean lattice gauge theory simulations that for not too large values of β the full SU(2) group is extremely well approximated by its 120 element subgroup Y, the symmetry group of the icosaheder.[7] At some large value of β the icosahedral group ceases to reproduce SU(2), because the discrete nature of the group forces a freezing transition.

With finite number of group elements the calculation is extremely simplified because both the multiplication table and the action can be tabulated. In order to test the region where the approximation by the discrete subgroup can be trusted we have performed Monte Carlo simulation of Z_1 using both the full SU(2) group and the icosahedral subgroup. The average action as function of β is shown in Fig. 5. From this figure it is clear that the freezing transition occurs around $\beta_F=2.9$. Below this point there is excellent agreement between the full SU(2) group and its discrete icosahedral subgroup.

On the same figure we display also the average action of the heat-kernel problem, A_2. This one is calculated by using only the icosahedral subgroup. We see that the two different actions are very similar for $\beta \geq 1.6$, well below β_F. We trust the calculation for $\beta<\beta_F$ as being representative of the full SU(2) group.

The 3-dimensional SU(2) chiral model is expected to have a continuous phase transition.[10] Our Monte-Carlo calculation on a 5^3 lattice with 1000 iterations per point has shown no hysteresis loop, thus supporting the expectation of a continuous phase transition. The fact that a phase transition does occur can be seen from Fig. 3 where we use statistical averages defined in Z_2 to establish a connection between β and the coupling constant of the Hamiltonian of Eq. (1).

The action A_1 had been previously investigated numerically by Kogut et al.[11] Although both calculations indicate the existence of a continuous phase transition we find ourselves disagreeing with the detailed functional behaviour of $\langle A_1 \rangle$ reported by them.

APPENDIX B: AN SU(2) SUPPLEMENT

Let us represent the group element G as a 2×2 matrix (the fundamental representation)

$$G = g_0 \mathbb{1} + i\vec{g}\cdot\vec{\sigma} \tag{B 1}$$

The four numbers are subject to the condition

$$\det G = g_0^2 + \vec{g}^2 = g_\alpha g_\alpha = 1$$

thus establishing the group manifold as the unit sphere in four dimensions S^3.

Using the four-dimensional language we note that any similarity transformation $G \to G_1^{-1} G G_1$ leaves trG invariant and, therefore, is equivalent to a rotation of the 3-vector $\vec{g}$

leaving g_0 unchanged. Under such a rotation G itself decomposes therefore into a scalar and a vector. Higher representations of the group will have higher rank tensor components under such similarity transformations. In fact the highest rank tensor in the representation j should be of order 2j following the simple sum-rule

$$\sum_{k=0}^{2j} (2k+1) = (2j+1)^2 \tag{B3}$$

For our purposes it suffices to consider just the scalar and vector parts of all representations. The scalar part is otherwise known as the character of the representation

$$\chi_j(G) = \sum_m \mathcal{D}^{(j)}_{mm}(G) \tag{B4}$$

It can depend only on the value g_0 and, in fact, has to be a polynomial of rank 2j in g_0. Using the convenient notation $g_0 = \cos\alpha$ it is easy to establish that

$$\chi_j(G) = \frac{\sin(2j+1)\alpha}{\sin\alpha} \tag{B5}$$

To find the linear combination of the matrix-elements of the j-th representation which behaves like a vector let us take its trace with the matrices $\vec{J}^{(j)}$ which define the generators of SU(2) in the j-th representation:

$$\sum_{mn} \mathcal{D}^{(j)}_{mn}(G)\, \vec{J}^{(j)}_{nm} = i\vec{g}\, \rho_j(G) \tag{B6}$$

The vector character is exhibited explicitly by $\vec{g}$ on the right-hand side. ρ_j has therefore to be a class-function, i.e. a scalar in our terminology. In fact it has to be a polynomical of degree 2j-1 in g_0.

To establish the general form of ρ_j let us note that the Casimir of SU(2) can be written as the transverse part of the Laplacian in the four-dimensional space spanned by g_α:

$$\Delta = \frac{1}{4}\left[\left(g_\alpha \frac{\partial}{\partial g_\alpha}\right)^2 + 2 g_\alpha \frac{\partial}{\partial g_\alpha} - g_\alpha g_\alpha \frac{\partial}{\partial g_\beta}\frac{\partial}{\partial g_\beta}\right] \tag{B7}$$

Requiring that both expressions (B4) and (B6) are eigenfunctions of Δ with eigenvalue $j(j+1)$ and using the constraint (B2) we find the differential equations

$$\left[(g_0^2-1)\frac{d^2}{dg_0^2} + 3g_0\frac{d}{dg_0} - 4j(j+1)\right]\chi_j = 0 \tag{B8}$$

$$\left[(g_0^2-1)\frac{d^2}{dg_0^2} + 5g_0\frac{d}{dg_0} + 3 - 4j(j+1)\right]\rho_j = 0 \tag{B9}$$

This establishes that χ_j and ρ_j are the Gegenbauer polynomials $C^1_{2j}(g_0)$ and $C^2_{2j-1}(g_0)$ respectively.[12] In fact

$$\rho_j = \frac{1}{2}\frac{d\chi_j}{d\cos\alpha} = \frac{(j+1)\sin 2j\alpha - j\sin 2(j+1)\alpha}{2\sin^3\alpha} \tag{B10}$$

The construction (B6) can be generalized to produce the higher rank tensors by using symmetric permutations of the generators of the group. However in our SU(2) problem, with the particular ansatz for the wave function in terms of group characters only, we need only the scalars χ_j and vectors $i\vec{g}\rho_j$.

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FIGURE CAPTIONS

Fig. 1: Ground-state energy for the mean-plaquette ansatz (full circles) compared with the result for a one-plaquette problem (open circles).

Fig. 2: Link variables (U_ℓ) and gauge-group variables (G_i) used in Eq. (17) to (21).

Fig. 3: The parameter β of the mean-link ansatz which minimizes the vacuum energy.

Fig. 4: Comparison of the ground-state energies of our two trial functions. The mean-link data stop at β_F. Extrapolating from lower β we expect them to follow the trend of the mean-plaquette data for lower g^2.

Fig. 5: Monte-Carlo evaluation of chiral SU(2) problems in 3 dimensions. A_1 is the fundamental- representation action of Eq. (A-1) and A_2 is the heat-kernel action of Eq. (A-6). A_1 is calculated for both SU(2) and its icosahedral subgroup $\tilde{Y}$ exhibiting a difference between the two above $\beta_F \simeq 2.9$.

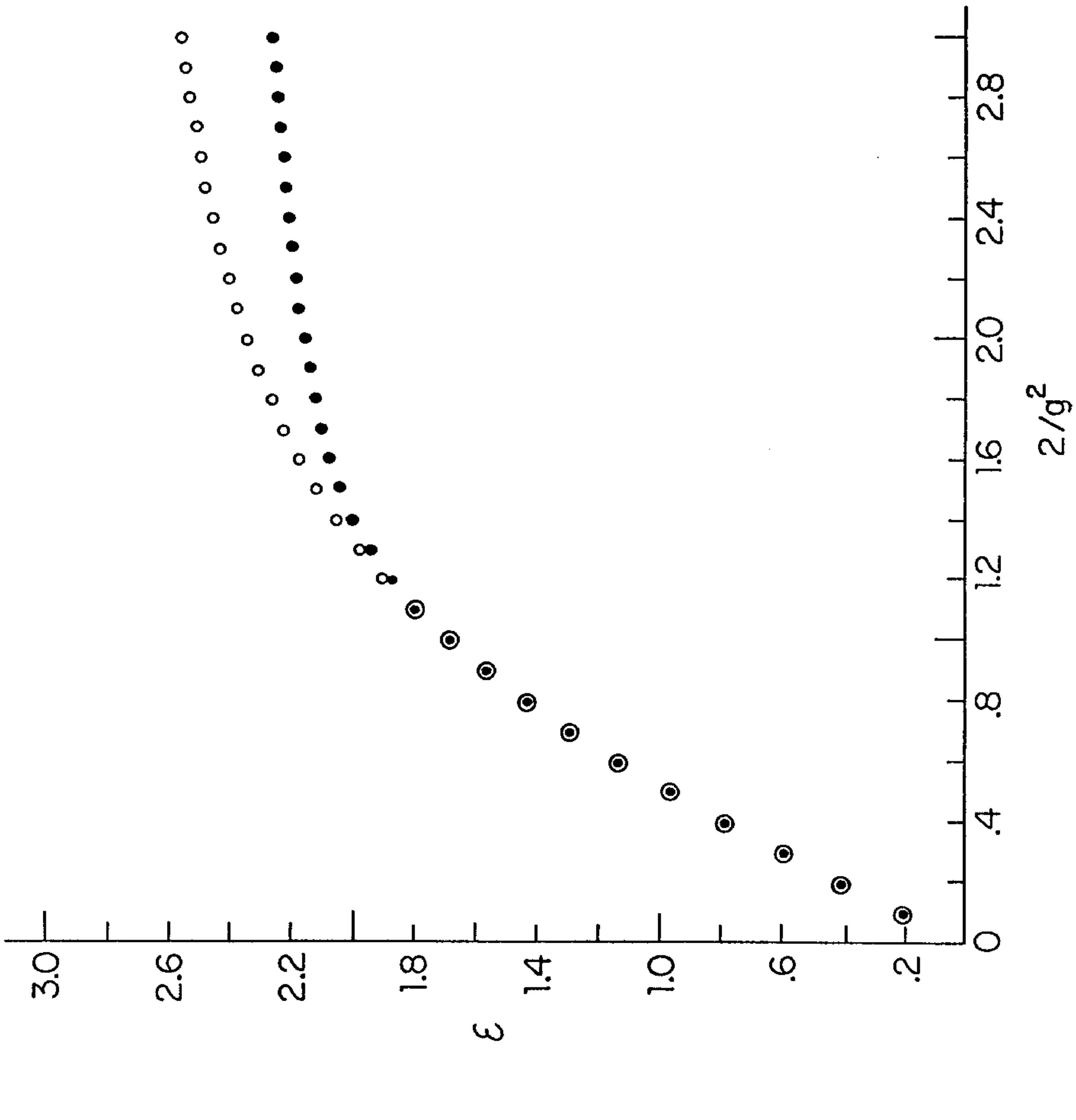


Fig. 1

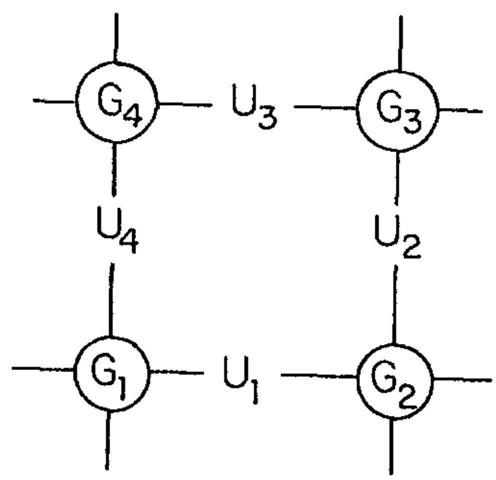


Fig. 2

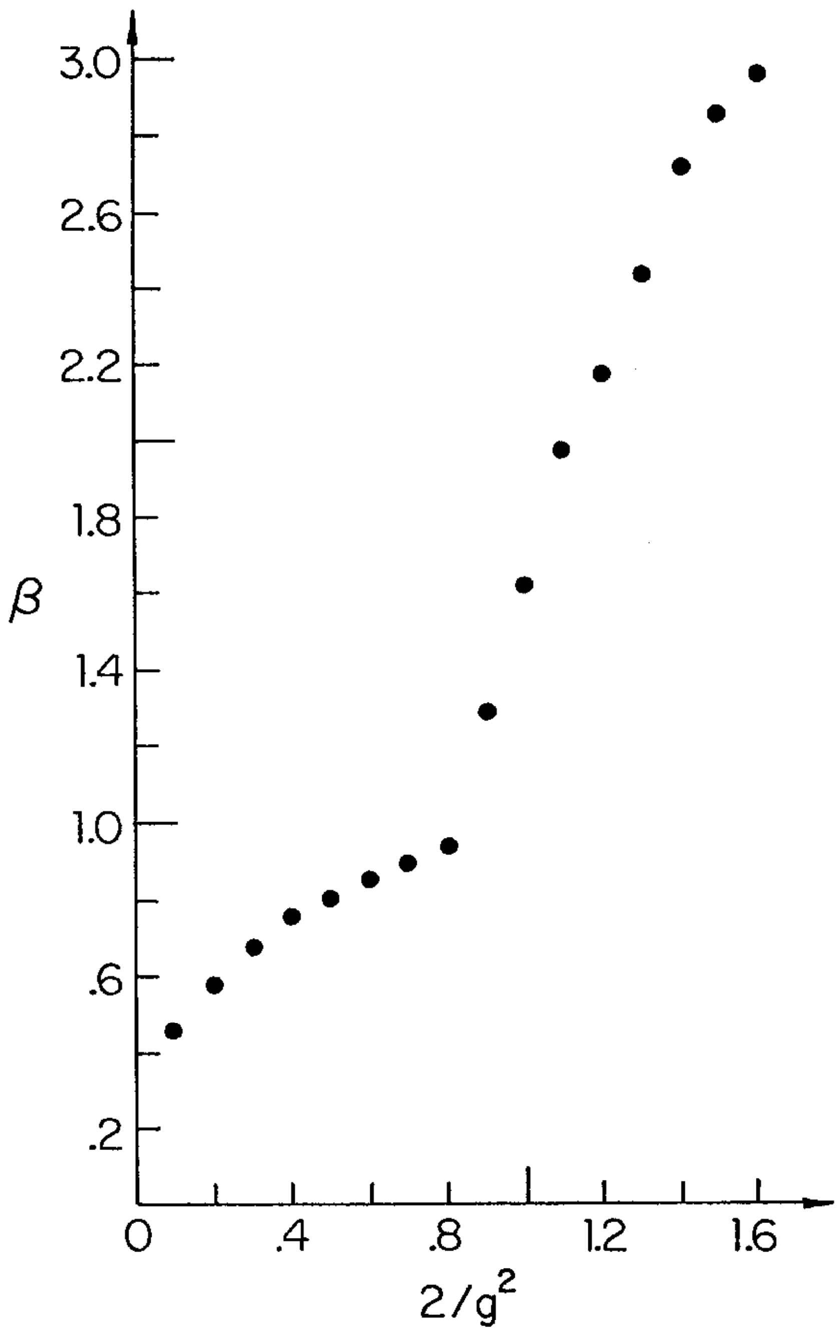


Fig. 3

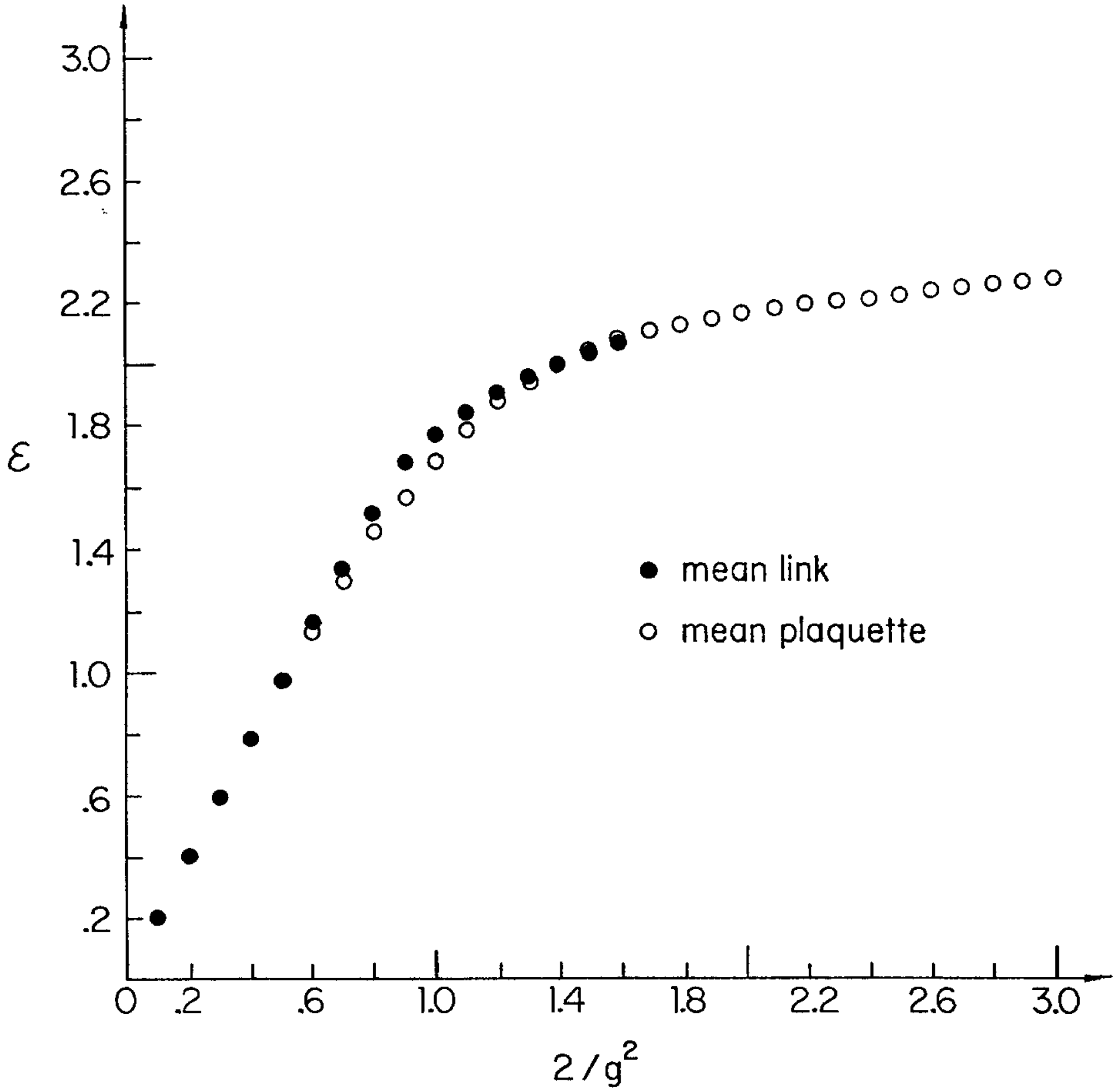
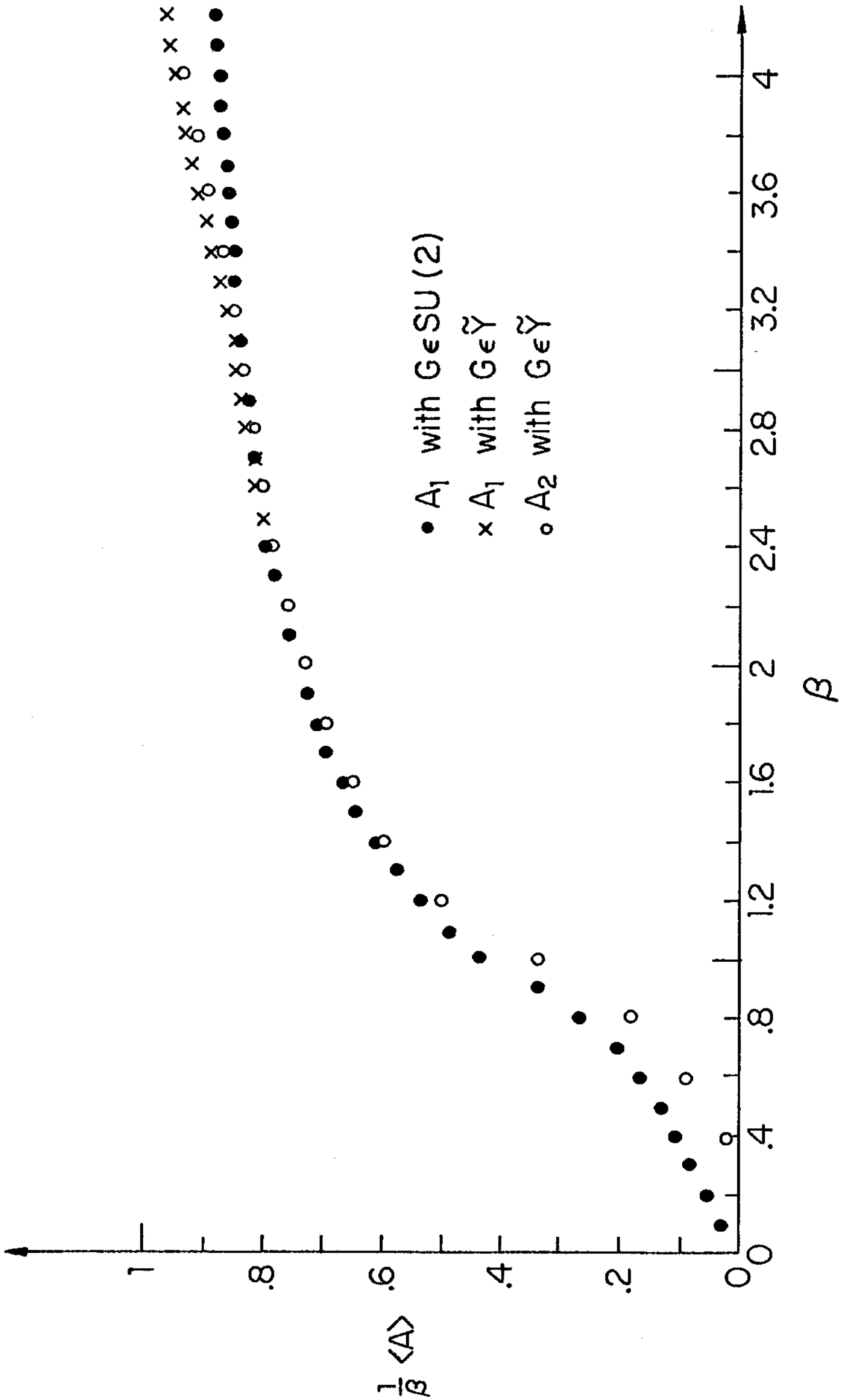

3.0
2.6
2.2
1.8
1.4
1.0
.6
.2
ε
mean link
mean plaquette
0
.2
.6
1.0
1.4
1.8
2.2
2.6
3.0
2 / g2

Fig. 4



• A_1 with $G \in SU(2)$
× A_1 with $G \in \tilde{Y}$
∘ A_2 with $G \in \tilde{Y}$
β
$\frac{1}{\beta}\langle A \rangle$
0
.4
.8
1.2
1.6
2
2.4
2.8
3.2
3.6
4
.2
.6
1

Fig. 5